Maxima and Minima

1. A main railway runs North-South, passing through a city B. A factory C is b km south of B and a km west of the railway. A side railway is constructed from C to the main railway in order to transport cargoes from B to C. Suppose the fee of transportation along the side and the main railway are $\$p$ / km / ton and $\$q$ / km / ton respectively ($p > q$). Find the angle between the side railway so that the transportation fee is minimum.
Give an interpretation in the theory of waves.

2. If y is given as a function of x by the equation $4x^2 + 6xy + 9y^2 - 8x - 24y + 4 = 0$, find the maximum and minimum values of y.

3. Show how to find and how to discriminate between the maximum and minimum values of a function of a single variable. A tree trunk, in the form of a frustum of a cone, is h cm long, and the greatest and least diameters are a and b cm respectively. A beam of square cross-section is cut from the tree; show that if $2a > 3b$, then the beam has a maximum volume when its length is $\dfrac{ha}{3(a-b)}$.
What is the length of the beam for maximum volume when $2a < 3b$?

4. The illumination of an area by a source of light is proportional to $\dfrac{x}{\sqrt{x^2+a^2}} - \dfrac{x}{\sqrt{x^2+b^2}}$, where a and b ($b > a$) are constants and x can be varied. Find the value of x which gives maximum illumination.

5. Find the semi-vertical angle of a right conical shell of minimum surface area which, with its rim resting on a table, can cover a sphere of radius r, and show that its area is $\left(3+2\sqrt{2}\right)\pi r^2$.

6. A cylindrical hole is drilled through a hemispherical casing of radius R, the axis of the hole being along the axis of the hemisphere. Find the length of the cylindrical wall of the hole when its volume is maximum, and determine the metal remaining in this case.

7. A tapering log has a square section whose side varies uniformly from a at one end to b ($b > \dfrac{3a}{2}$) at the other. Show that the volume of the greatest circular cylinder that can be cut from the log is $\dfrac{\pi b^3 L}{27(b-a)}$, where L is the length of the log.

8. A number of thin sheet-metal cylindrical cans with lids are required. Each can is to hold 75 cubic cm, and the lid is to lap closely over the top of the can by $\dfrac{1}{2}$ cm. If the amount of metal to be used is to be a minimum, find an equation to determine the radius of a can.

9. Prove that, if $f'(a) = 0$ and $f''(a) > 0$, $f(x)$ has a minimum value for $x = a$. Find all the values of θ

between 0 and $\frac{1}{2}\pi$ for which $\cos 9\theta \sec^2 \theta$ is a minimum.

10. Prove that $x + \dfrac{x^3}{3} < \tan x$ for $x \in \left(0, \dfrac{\pi}{2}\right)$.

11. Prove that $\left(1 - \dfrac{\theta^2}{2}\right)\cos\theta + \theta\sin\theta$ increases as θ increases so long as θ is positive and less than π.

12. Prove that, if $x > 0$, $x - \dfrac{x^3}{3} < \tan^{-1} x < x - \dfrac{x^3}{3} + \dfrac{x^5}{5}$.

13. Prove that $e^{-x} - 1 + x$ is never negative.

If $f(x) = \dfrac{e^x - 1}{x}$ and $x \neq 0$ and $f(0) = 1$, prove that $f(x)$ is an increasing function of x for all x.

Sketch the graph of $f(x)$.

14. Establish the inequalities:

(i) $\tan x > x$,

(ii) $1 < \dfrac{x}{\sin x} < \dfrac{\pi}{2}$ in the range $0 < x < \dfrac{\pi}{2}$.

15. **(a)** Locate the minima and maxima of $f: [-2, +\infty) \to \mathbf{R}$, $f(x) = 2|x|^3 - 9x^2 + 12|x| + 1$

(b) Show that $\cos\left(\dfrac{\pi x}{2}\right) < 1 - x^2$ for $0 < x < 1$.

16. **(a)** Show, for $t > 0$, $0 < p < 1$, $(1 + t)^p > 1 + t^p$.

(b) Show that for $x, y > 0$, $0 < p < 1$, $(x + y)^p > x^p + y^p$.

(c) How about when $p > 1$? $p < 0$?

17. Find the absolute maximum and minimum of $|x|^3 - 6x^2 + 11|x| - 6$.

18. Find the maxima, minima and points of inflection of the following functions :

(a) $x^3 - 6x + 2$ **(b)** $\dfrac{x^3}{x^4 + 1}$ **(c)** $\dfrac{2x}{1 + x^2}$

19. Show that the minima and minima of $\sin mx \csc x$ ($m \in \mathbf{Z}$) are given by $\tan mx = m \tan x$;
and deduce that $\sin^2 mx \leq m^2 \sin^2 x$.

(Observe that $\dfrac{\sin^2 mx}{\sin^2 x} = m^2 \dfrac{\cos^2 mx}{\cos^2 x} = m^2 \dfrac{1 + \tan^2 x}{1 + \tan^2 mx} = m^2 \dfrac{1 + \tan^2 x}{1 + m^2 \tan^2 x}$ at the minimum or maximum.)

20. Show that the greatest value of $x^m y^n$, where $x, y > 0$ and $x + y = k$ $(m, n, k > 0)$ is $\dfrac{m^m n^n k^{m+n}}{(m+n)^{m+n}}$.

21. Given n real numbers $a_1, a_2, \ldots, a_n$, determine the value of x for which the sum $\sum_{k=1}^{n}(x-a_k)^2$ is a minimum. (You must show that this is a minimum, not merely a minimum or a maximum)

22. Determine the constants A and B so that the curve $y = Ax^{1/2} + Bx^{-1/2}$ will have an inflectional point $(4, 6)$.

23. Show that the curve $(1 + x^2)\, y = 1 - x$ has three points of inflection and that they lie on a straight line.

24. Let $y = f(x)$ be a differentiable function satisfying $f(a+b) = \dfrac{f(a)+f(b)}{1-f(a)f(b)}$, $f(0) = 0$ and $f'(0) = 1$, prove that $f(x)$ is convex for all $x > 0$.

25. **(a)** Let $f_1(x) = \sqrt{(x-a)^2 + b^2}$, $f_2(x) = \sqrt{(x-c)^2 + d^2}$ where $a > c$.

Set $f(x) = \text{Max}\,\{f_1(x), f_2(x)\}$

(i) Find x_0 such that $f(x) = \begin{cases} f_1(x) & \text{if } x < x_0 \\ f_2(x) & \text{if } x > x_0 \end{cases}$

(ii) Find x_1 such that $f(x_1) \le f(x)$ for all x.

(b) Making use of **(a)** or otherwise, solve the following problem: Two persons are at points p_1, p_2 separated by a straight line L on a place. They went to see each other at some point on L and agree that whoever arrives there first should wait for the other. Assuming that they start at the same time and walk at the same speed, find the point p on L so that they can meet within the shortest time.

26. **(a)** Prove that for any $x \in (0, \pi)$, $$\sum_{k=1}^{n}\cos kx = \frac{\sin\left[\left(n+\frac{1}{2}\right)x\right] - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$$

(b) For any positive integer n, let $S_n(x) = \sum_{k=1}^{n}\dfrac{\sin kx}{k}$, $x \in [0, \pi]$.

(i) Prove that if $x_0 \in (0, \pi)$ and $S_n'(x_0) = 0$, then $\sin\left[\left(n+\frac{1}{2}\right)x_0\right] - \sin\frac{1}{2}x_0 = 0$

and hence deduce that $\sin nx_0 \ge 0$.

(ii) It is given that for each n, $S_n(x)$ attains its absolute minimum on $x \in [0, \pi]$.
If $S_{m+1}(x)$ attains its absolute minimum at some $x_0 \in (0, \pi)$, show that $S_{m+1}(x_0) \le 0$.
Hence show that if $S_m(x) > 0$ for all $x \in (0, \pi)$, then $S_{m+1}(x)$ attains its absolute minimum only at $x = 0$ and $x = \pi$.

(iii) Prove by induction that for any positive integer n, $S_n(x) > 0$ for all $x \in (0, \pi)$.

27. $F(x) = \sec x - x - 1$ be a function defined on $\left[0, \frac{\pi}{2}\right)$ and let $\lambda \in \left[0, \frac{\pi}{2}\right)$ such that $F'(\lambda) = 0$.

(a) Prove that

(i) $\lambda = \sin^{-1}\left[\frac{1}{2}\left(\sqrt{5} - 1\right)\right]$

(ii) F is strictly decreasing on $[0, \lambda]$ and strictly increasing on $\left[\lambda, \frac{\pi}{2}\right)$.

(i.e. if $0 < x_1 < x_2 < \lambda$, then $F(x_1) > F(x_2)$ and if $\lambda \le x_1 < x_2 < \frac{1}{2}\pi$, then $F(x_1) < F(x_2)$.)

(iii) F attains its absolute minimum at $x = \lambda$, with $F(\lambda) < 0$.

(b) Let $x_1, x_2, \ldots$ be the sequence in $\left[0, \frac{\pi}{2}\right)$ defined by $x_1 = \lambda$ and

$x_{n+1} = \sec x_n - 1 \qquad (n = 1, 2, \ldots)$

Prove that $x_{n+1} < x_n$ for all positive integers n .

Suppose $\lim_{n\to\infty} x_n = \mu$. Prove that

(i) $0 < \mu < \lambda$

(ii) $F(\mu) = 0$ and hence show that $\mu = 0$.

28. The intensity of illumination at any point varies inversely as the square of the distance between the point and the light source. Two lights, one having intensity eight times that of the other, are 6 m apart. Find a point between the two lights at which the total illumination is a minimum.

29. Two ships are sailing uniformly with velocities u, v along straight lines inclined at an angle θ . Show that if a, b be their distances at one time from the point of intersection of the course, then the least distance of the ships is equal to $\dfrac{(av - bu)\sin\theta}{\sqrt{u^2 + v^2 - 2uv\cos\theta}}$.

30. An electric bulb is placed directly above the center of a round table of radius r . How high above the table should the electric bulb be placed in order that the surface of the table near the edge may be best illuminated? It is known that the intensity of illumination on an area is inversely proportional to the product of the square of the distance from the source of light and the secant of the angle of incidence.